Homogenization of a Hele-Shaw-type problem in periodic time-dependent media

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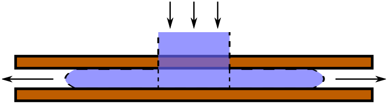
Hele-Shaw problem

Model of the $\underbrace{\text{pressure-driven}}_{\vec{v}=-Du}$ flow of $\underbrace{\text{incompressible liquid}}_{\text{div }\vec{v}=0}$ in

- **Hele-Shaw cell**: two parallel plates close to each other



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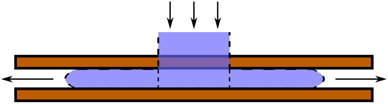
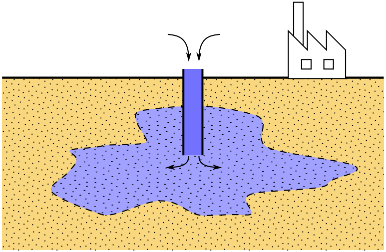


- **porous medium**



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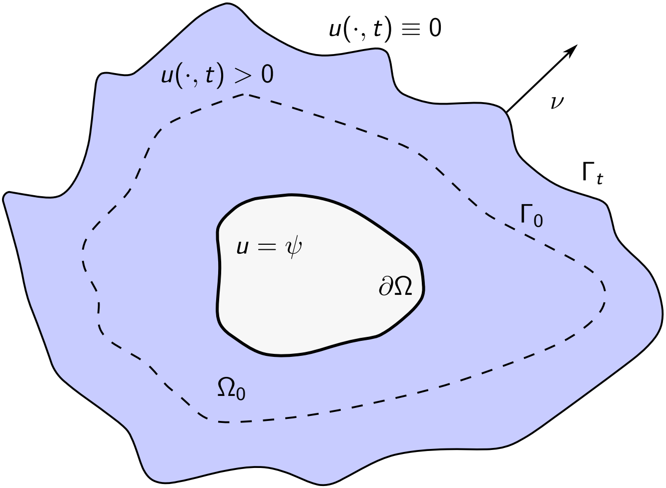
- find $u : \overline{Q} \to [0, \infty)$ satisfying formally

$$\begin{cases} -\Delta_x u(x,t) = 0 & \text{in } \{u > 0\} \cap Q \\ V_\nu(x,t) = g(x,t)\,|D_x u(x,t)| & \text{on } \partial\{u > 0\} \cap Q \end{cases}$$

Hele-Shaw problem

- **(Initial data)** wet region $\{u > 0\} = \Omega_0$ at $t = 0$
- **(Boundary data)** $u(x,t) = \psi(x,t) > 0$ on $\partial\Omega$



Hele-Shaw problem: homogenization

- find $u^\varepsilon : \overline{Q} \to [0, \infty)$ satisfying formally

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What is it?

Assumptions on g

For existence of solutions:

- **(regularity)**

$$g \in \mathrm{Lip}(\mathbb{R}^n \times \mathbb{R})$$

- **(non-degeneracy)** there exist constants m, M such that

$$0 < m \leq g(x,t) \leq M \qquad \forall (x,t) \in \mathbb{R}^n \times \mathbb{R}$$

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To see averaging as $\varepsilon \to 0$:

- **(periodicity)** g is $\mathbb{Z}^{n+1}$-periodic, i.e.,

$$g(x+k, t+l) = g(x,t) \qquad \forall (x,t) \in \mathbb{R}^n \times \mathbb{R}, \forall (k,l) \in \mathbb{Z}^n \times \mathbb{Z}$$

Homogenized problem of Hele-Shaw

If the problem homogenizes, u^ε should converge in some sense to the solution of

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Is the problem well-posed?

Well-posedness of homogenized problem

Assume that $r(q)$ satisfies:

- **(non-degeneracy)** there exist constants m, M such that $0 < m \leq M$ such that

$$m\,|q| \leq r(q) \leq M\,|q| \qquad \forall q \in \mathbb{R}^n$$

- **(ellipticity)**

$$r^*(q) \leq r_*(aq) \qquad q \in \mathbb{R}^n,\ a > 1$$

Well-posedness of homogenized problem

Theorem (P. '12')

Let $f(x,t,q) = g(x,t)\,|q|$ or $f(x,t,q) = r(q)$. Then the Hele-Shaw-type problem

$$\begin{cases} -\Delta u = 0 & \text{in } \{u > 0\} \cap Q \\ V_\nu = f(x,t,Du) & \text{on } \partial\{u > 0\} \cap Q \end{cases}$$

has unique viscosity solution for any sufficiently regular initial and boundary data.

extends the previous results by Kim '04,'07, using ideas from Kim & P. '12

Homogenization result

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Suppose that $g(x, t)$ is positive, Lipschitz, Z^{n+1}-periodic and that initial and boundary data are regular so that well-posedness theorem applies.

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Suppose that $g(x,t)$ is positive, Lipschitz, Z^{n+1}-periodic and that initial and boundary data are regular so that well-posedness theorem applies.

Then there exists $r(q) : \mathbb{R}^n \to [0,\infty)$ that is (non-degenerate) and (elliptic) such that the solutions u^ε of

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with given boundary/initial data converge as $\varepsilon \to 0$ in the sense of half-relaxed limits to the solution u of

$$\begin{cases} -\Delta u = 0 & \text{in } \{u > 0\} \\ V_\nu = r(Du) & \text{on } \partial\{u > 0\} \end{cases}$$

with the same boundary data.

Homogenized velocity $r(q)$

What is the form of $r(q)$?

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g **independent of time**: $g(x, t) = g(x)$

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$$r(q) = \underbrace{\frac{1}{\left\langle \frac{1}{g} \right\rangle}}_{\text{constant}} |q| \qquad \left\langle \frac{1}{g} \right\rangle = \int_{[0,1]^n} \frac{1}{g(x)}\, dx$$

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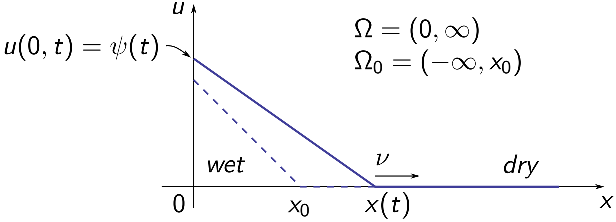
$$\begin{cases} -\Delta u^\varepsilon = 0 \\ V_\nu = g\left(\frac{x}{\varepsilon}\right) |Du^\varepsilon| \end{cases} \qquad \xrightarrow{\varepsilon \to 0} \qquad \begin{cases} -\Delta u = 0 \\ V_\nu = \frac{1}{\left\langle \frac{1}{g} \right\rangle} |Du| \end{cases}$$

one dimensional problem: $n = 1$



Homogenized velocity $r(q)$ in 1D

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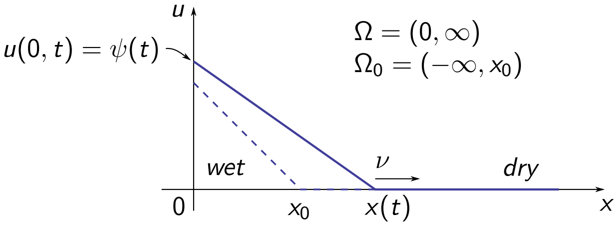

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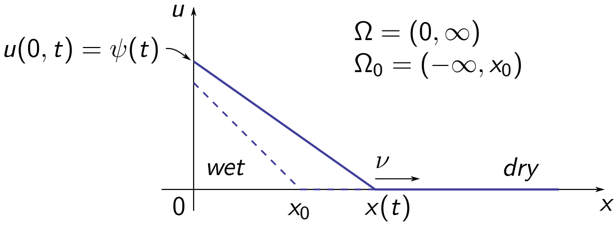

- harmonic functions are linear $\Rightarrow |Du|$ is given by $\psi(t)/x(t)$
- Hele-Shaw problem reduces to an ODE for the position $x^\varepsilon(t)$ of the free boundary (a point).

$$\dot{x}^\varepsilon(t) = g\left(\frac{x^\varepsilon(t)}{\varepsilon}, \frac{t}{\varepsilon}\right) \frac{\psi(t)}{x^\varepsilon(t)}, \qquad x^\varepsilon(0) = x_0$$

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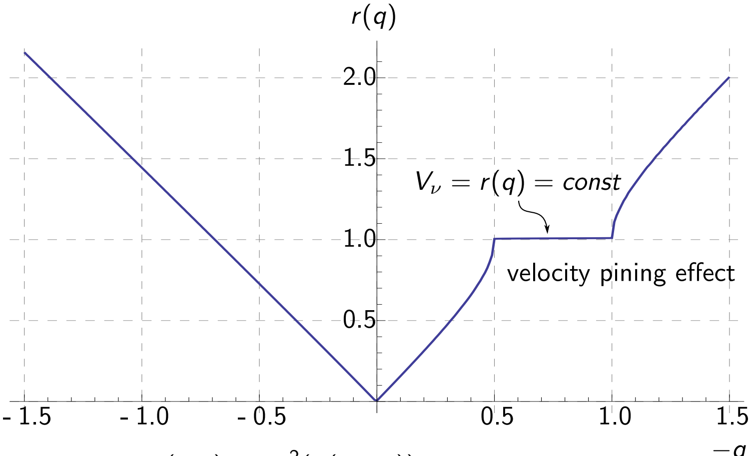
- no explicit formula for $r(q)$, only estimate $|q| \min g \leq r(q) \leq |q| \max g$

Homogenized velocity $r(q)$: example



$g(x,t) = \sin^2(\pi(x-t)) + 1$

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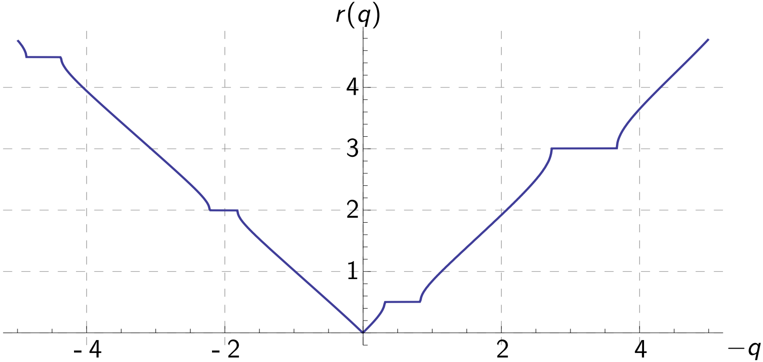

$g(x, t) = sin(2\pi(x - 3t))sin(2\pi(x + 2t)) + 1.1$

Identification of $r(q)$

Standard homogenization in periodic setting:

- **Hamilton-Jacobi equations**: Lions, Papanicolau, Varadhan '87 (unpublished), Evans '91
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$$\forall P \in \mathbb{R}^n \quad \exists \text{periodic } v,\ \exists! \overline{H}(P) \qquad H(Dv + P, x) = \overline{H}(P)$$

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This approach does not apply to the Hele-Shaw problem!

Identification of $r(q)$

Idea: use an **obstacle problem**

- **elliptic equations** (random): Caffarelli, Souganidis & Wang '05
- **Hele-Shaw**, **contact angle dynamics** (periodic): Kim '07–

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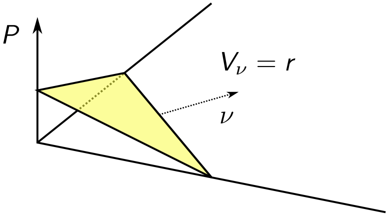
- (HP) has traveling wave solutions: for $q \in \mathbb{R}^n \setminus \{0\}$, $r > 0$

$$P_{q,r}(x,t) = |q|\,(rt - x \cdot \nu)_+, \qquad \nu = \frac{-q}{|q|}$$



- $P_{q,r}$ is a solution if and only if $r = r(q)$

Identification of $r(q)$

- For given $q \neq 0$, find a solution $u_{\varepsilon;q}$ on $\mathbb{R}^n \times [0,\infty)$ of the ε-problem for every $\varepsilon > 0$ with initial data

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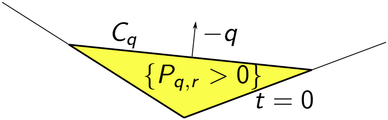
What does slower and faster mean?

Obstacle problem

Use $P_{q,r}$ as an obstacle:

- Domain: $Q_q = C_q \times [0, \infty)$
  C_q ... cylinder with axis in the direction $-q$



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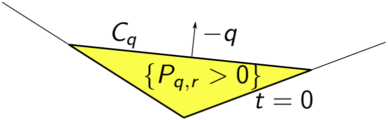

- Solution:

$$\overline{u}_{\varepsilon;q,r} = \sup \{v : \text{subsolution of } \varepsilon\text{-problem on } Q_q,\ v \leq P_{q,r}\}$$

$$\underline{u}_{\varepsilon;q,r} = \inf \{v : \text{supersolution of } \varepsilon\text{-problem on } Q_q,\ v \geq P_{q,r}\}$$

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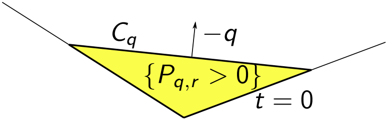

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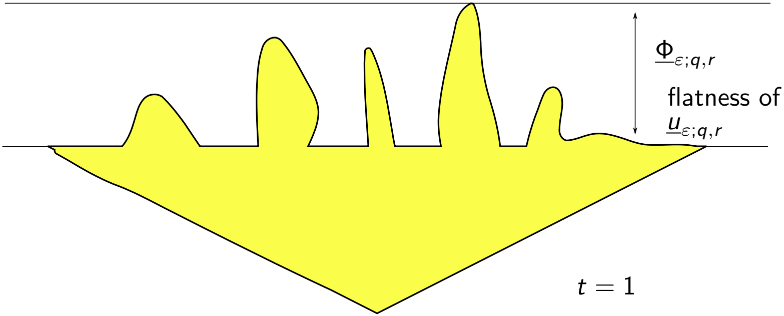
- Nice properties:
  - $\overline{u}_{\varepsilon;q,r}$ ... subsolution, $\underline{u}_{\varepsilon;q,r}$ ... supersolution
  - solutions when not touching the obstacle
  - $\overline{u}_{\varepsilon;q,r} = \underline{u}_{\varepsilon;q,r} = P_{q,r}$ on the boundary ∂Q_q

Flatness

We introduce a new quantity: **flatness of the solution**

- measures how much the solution obstacle problem detaches from the obstacle
- indicator of how good our guess of r is for a given slope



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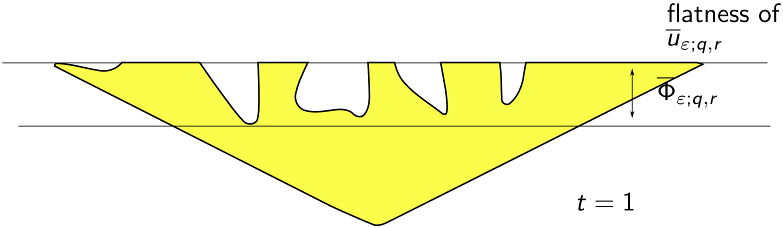
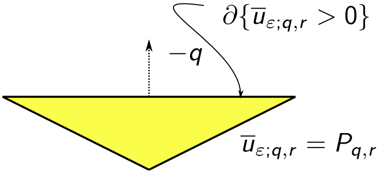


Candidates for $r(q)$



$t = 0$

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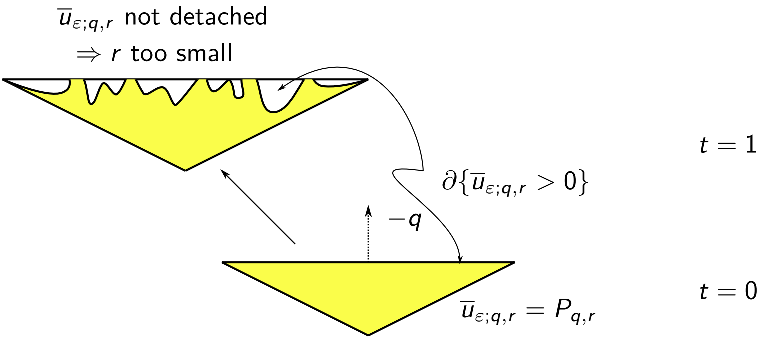

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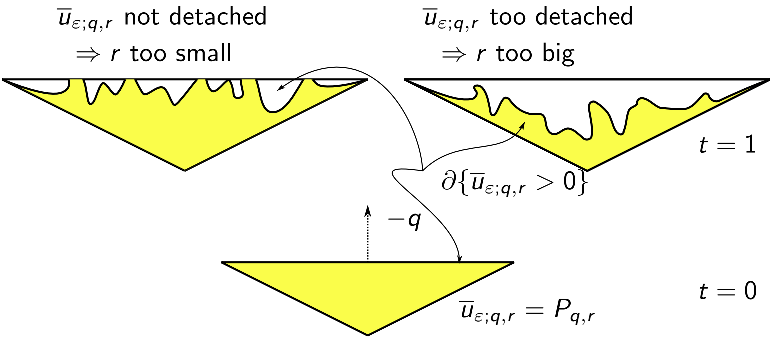

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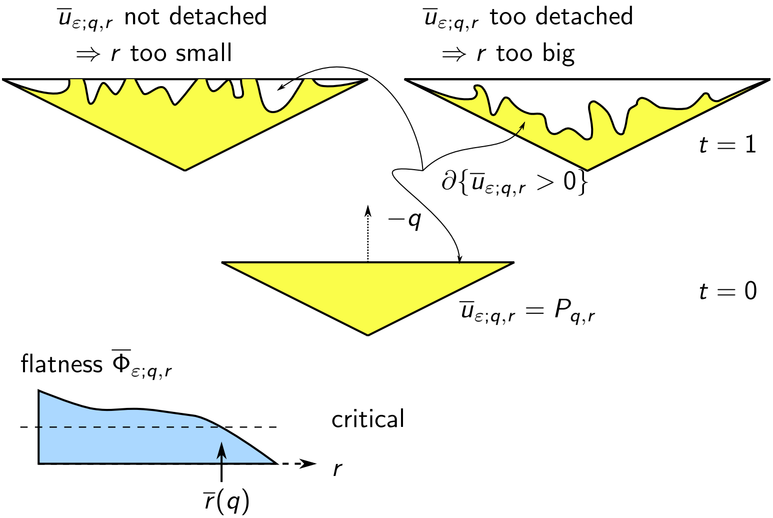

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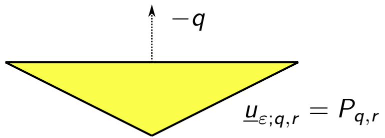

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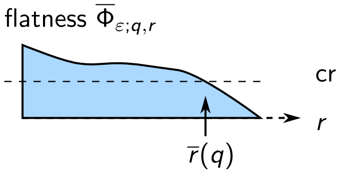

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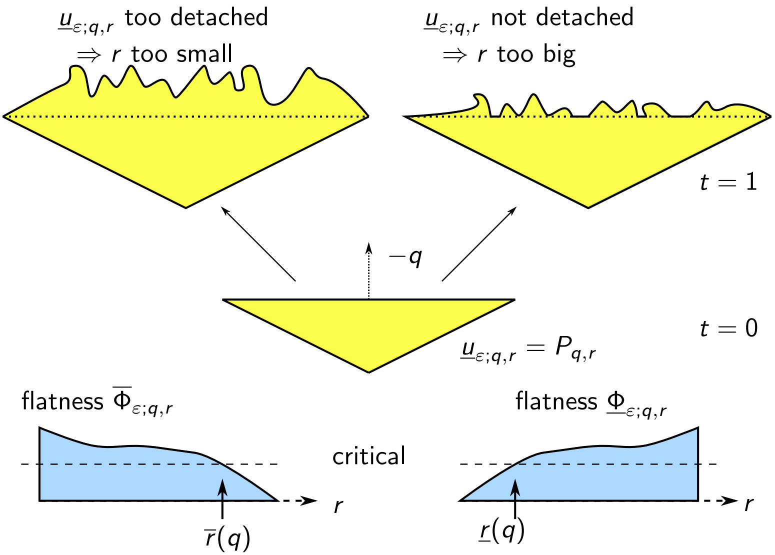

Cone flatness

Can the free boundaries of $\overline{u}_{\varepsilon;q,r}$ or $\underline{u}_{\varepsilon;q,r}$ have long thin fingers?

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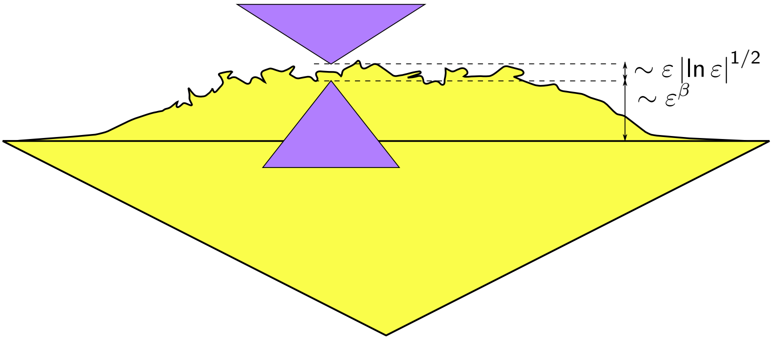
Lemma (P. '12)

There exists $K > 0$, a constant independent of ε, such that for $\varepsilon > 0$ small the free boundaries of $\overline{u}_{\varepsilon;q,r}$ and $\underline{u}_{\varepsilon;q,r}$ are in between cones $K\varepsilon \left|\ln \varepsilon\right|^{1/2}$ apart.

Cone flatness



Local comparison principle

We can compare solutions far away from the boundary for short time even if the boundary data is not ordered.

Lemma (Kim '07, P. '12)

Let $\beta \in (4/5, 1]$. Suppose that $r_1 > r_2 > 0$, $a > 1$ and $q \neq 0$ and that ε is sufficiently small. Then

$$\overline{u}_{\varepsilon;q,r_1} \qquad \text{and} \qquad \underline{u}_{\varepsilon;aq,r_2}$$

cannot be both ε^β-flat.

Critical flatness ε^β

The critical value of flatness is

$$\overline{\Phi}_{\varepsilon;q,r} \sim \varepsilon^\beta \sim \underline{\Phi}_{\varepsilon;q,r}$$

for some fixed

$$\beta \in (4/5, 1).$$

Candidates for $r(q)$

Flatness provides **two** candidates for the homogenized velocity $r(q)$ for any $q \neq 0$:

- upper velocity

$$\underline{r}(q) = \sup \left\{ r > 0 : \limsup_{\varepsilon \to 0} \varepsilon^{-\beta} \underline{\Phi}_{\varepsilon;q,r} \geq 1 \right\}$$

- lower velocity

$$\overline{r}(q) = \inf \left\{ r > 0 : \limsup_{\varepsilon \to 0} \varepsilon^{-\beta} \overline{\Phi}_{\varepsilon;q,r} \geq 1 \right\}$$

Tools

Finally, prove that $\overline{r}(q)$ and $\underline{r}(q)$ have the desired properties using:

- scaling
- monotonicity (Birkhoff property)
- local comparison principle
- cone flatness

In particular,

- **(semi-continuity)**

$$\overline{r}_* = \underline{r}, \qquad \overline{r} = \underline{r}^*$$

- **(ellipticity)**

$$\overline{r}(q) \leq \underline{r}(aq) \qquad \forall q \in \mathbb{R}^n, a > 1$$

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We set $r(q) = \overline{r}(q)$. $\quad \Rightarrow u$ solves $\begin{cases} -\Delta u = 0 \\ V_\nu = r(Du) \end{cases}$

Open problems

- continuity, Hölder regularity of $r(q)$?
- rate of convergence
- random environments (spatial or spatio-temporal): open
- extension to non-monotone problems: Hele-Shaw with mean curvature, contact angle dynamics etc.

The end

Thank you!

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- $a \in (0, 1)$
- Hele-Shaw problem has natural hyperbolic scaling:

$$\underline{u}_{\varepsilon;q,r}(x,t) \mapsto a\underline{u}_{\varepsilon;q,r}\left(\frac{x}{a}, \frac{t}{a}\right)$$

  is solution of the obstacle problem with $\varepsilon' = a\varepsilon$ on $aQ_q \subset Q_q$.
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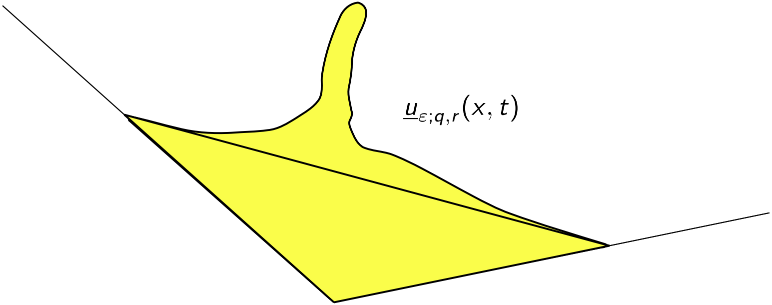
- By periodicity:

$$a\underline{u}_{\varepsilon;q,r}\left(\frac{x}{a}, \frac{t}{a}\right) \le \underline{u}_{a\varepsilon;q,r}(x-k, t-l) \qquad \text{for } (k,l) \in a\varepsilon(\mathbb{Z}^n \times \mathbb{Z})$$

  as long as $aQ_q \subset Q_q + (k,l)$ and the obstacles are ordered.

Monotonicity



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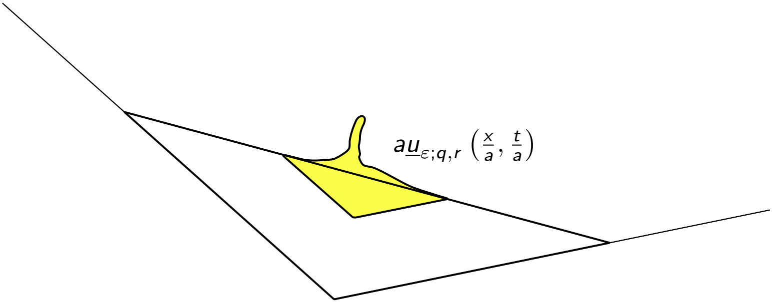

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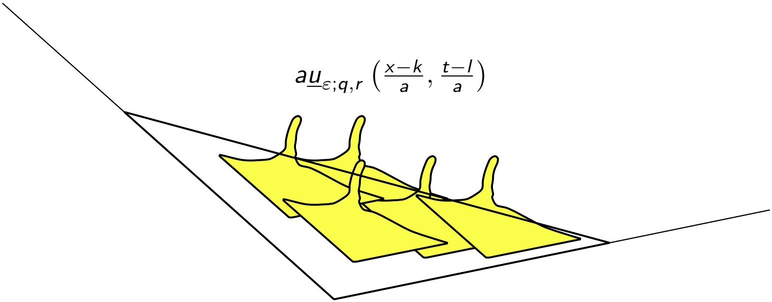

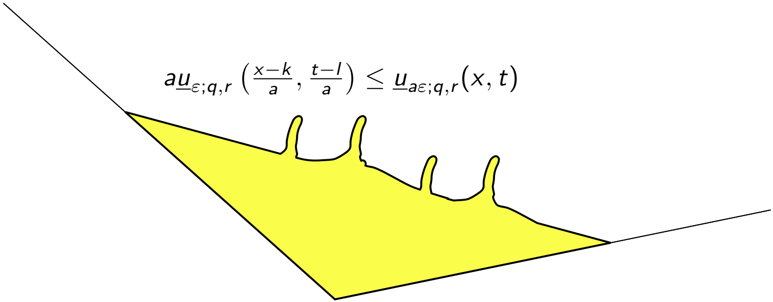

Monotonicity is also known as Birkhoff property.